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3 (Sem–5) MAT M 1

2021

(Held in 2022)

MATHEMATICS

(Major)

Paper : 5·1

(Real and Complex Analysis)

Full Marks : 60

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following as directed : 1×7=7

(a) Define limit of a function of two variables.

(b) Evaluate :

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+\left(x^2-y^2\right)^2}$$

Contd.

(c) Give an example of a function which is not continuous but Riemann integrable.

(d) When is an improper integral said to be convergent ?

(e) Find the fixed points of the transformation $w = \frac{2z+3}{z-4}$, z is a complex number.

(f) Let C_1 and C_2 be two simple closed curves, then $\oint_{C_1} z\,dz = \oint_{C_2} z\,dz$.

(State true **or** false)

(g) Verify whether the transformation $w = z^3$ is conformal or not at all points of the region $|z| < 1$.

2. Answer the following questions : 2×4=8

(a) Show that the following function is discontinuous at the origin :

$$f(x, y) = \begin{cases} \frac{1}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

(b) Show that a constant function k is integrable and

$$\int_a^b k\,dx = k(b-a)$$

(c) Test the convergence of

$$\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$$

(d) Evaluate : $\lim_{z \to i} \frac{z^{10}+1}{z^6+1}$

3. Answer ***any three*** parts : 5×3=15

(a) Show that $f(xy, z-2x) = 0$ satisfies, under suitable conditions, the equation $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2x$. What are these conditions ?

(b) Show that $\int_1^2 f\,dx = \frac{11}{2}$, where $f(x) = 3x+1$.

(c) Show that the integral

$$\int_0^{\pi/2} \frac{\sin^m x}{x^n} dx$$

exits, iff $n < m + 1$.

(d) Prove that the real and imaginary parts of an analytic function of a complex variable when expressed in polar form satisfy the equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

(e) Evaluate :

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz,$$ where C is the circle $|z| = 3$.

4. Answer ***any three*** parts : 10×3=30

(a) (i) Show that the function f, where

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & , \text{ if } x^2 + y^2 \neq 0 \\ 0 & , \text{ if } x = y = 0 \end{cases}$$

is continuous, possesses partial derivative, but is not differentiable at the origin. 6

(ii) Show that the function

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

is continuous at the origin. 4

(b) (i) Prove that $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}$ is invariant for change of rectangular axes. 6

(ii) Expand $x^2 y + 3y - 2$ in powers of $x - 1$ and $y + 2$. 4

(c) (i) If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, then find the maximum value of xyz. 5

(ii) Prove that the improper integral $\int_a^b f dx$ converges if and only if to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that

$$\left| \int_{a+\lambda_1}^{a+\lambda_2} f dx \right| < \varepsilon, \quad 0 < \lambda_1, \lambda_2 < \delta.$$ 5

(d) (i) The oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the sit $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ of numbers. 5

(ii) Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is integrable in $[0, 3]$ and $\int_0^3 [x]\,dx = 3$. 5

(e) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. Hence find v such that $f(z) = u + iv$ is analytic. 4+6=10

(f) (i) Evaluate :

$$\oint_C \frac{z^2}{(z-1)(z-2)}dz$$

and

$\oint_C \frac{e^{2z}}{(z+1)^4}dz$, where C is the circle $|z| = 3$. 3+2=5

(ii) Determine the region of the w plane into which each of the following is mapped by the transformation $w = z^2$: 2+3=5

(A) First quadrant of the z-plane

(B) Region bounded by $x = 1$, $y = 1$ and $x + y = 1$
