Total number of printed pages–11

3 (Sem–4/CBCS) MAT HC 3

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1 \times 10 = 10$

   (a) The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. What are the units of Z?

   (b) What is the trivial subring of R?

Contd.

2. Answer ***any five*** : 2×5=10

(a) Define ring. What is the unity of a polynomial ring $Z[x]$?

(b) Prove that in a ring R, $(-a)(-b)=ab$ for all $a, b \in R$.

(c) Prove that set S of all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with a and b, forms a sub-ring of the ring R of all 2×2 matrices having elements as integers.

(d) Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then prove that the characteristic of R is n.

(e) Let $z/4z = \{0+4z, 1+4z, 2+4z, 3+4z\}$. Find $(2+4z)+(3+4z)$ and $(2+4z)(3+4z)$.

(f) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in Z \right\}$ and let ϕ be the mapping defined as $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to a-b$.

Show that ϕ is a homomorphism.

(g) Let $f(x) = 4x^3 + 2x^2 + x + 3$ and $g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$ where $f(x), g(x) \in Z_5[x]$.

Compute $f(x)+g(x)$ and $f(x) \cdot g(x)$.

(h) Prove that in an integral domain, every prime is an irreducible.

3. Answer ***any four*** : 5×4=20

(a) Define a sub-ring. Prove that a non-empty subset S of a ring R is a sub-ring if S is closed under subtraction and multiplication, that is if $a-b$ and ab are in S whenever a and b are in S. 1+4=5

(b) Prove that the ring of Gaussian integers $Z[i] = [a + ib \mid a, b \in Z]$ is an integral domain.

(c) Let R be a commutative ring with unity and let A be an ideal of R. Then prove that R/A is an integral domain if and only if A is prime.

(d) If D is an integral domain, then prove that $D[x]$ is an integral domain.

(e) (i) If R is commutative ring then prove that $\phi(R)$ is commutative, where ϕ is an isomorphism on R. 3

(ii) If the ring R has a unity 1, $S \neq \{0\}$ and $\phi : R \to S$ is onto, then prove that $\phi(1)$ is the unity of S. 2

(f) Let $f(x) \in Z[x]$. If $f(x)$ is reducible over Q, then prove that it is reducible over Z.

(g) Consider the ring

$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} a, b \in Z \right\}$. Show that

$\phi : \mathbb{C} \to S$ is given by

$\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is a ring isomorphism.

(h) Prove that $Z[i] = \{a + bi \mid a, b \in Z\}$, the ring of Gaussian integers is an Euclidean domain.

4. Answer ***any four*** : 10×4=40

(a) (i) Prove that the set of all continuous real-valued functions of a real variable whose graphs pass through the point $(1, 0)$ is a commutative ring without unity under the operation of pointwise addition and multiplication [that is, the operations $(f + g)(a) = f(a) + g(a)$ and $(f.\, g)(a) = f(a) \cdot g(a)$. 6

(ii) Prove that if a ring has a unity, it is unique and if a ring element has an inverse, it is unique. 4

(b) Define a field. Is the set I of all integers a field with respect to ordinary addition and multiplication? Let

$Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$. Prove that

$Q[\sqrt{2}]$ is a field. 2+1+7=10

(c) (i) Prove that the intersection of any collection of subrings of a ring R is a sub-ring of R. 5

(ii) Let R be a commutative ring with unity and let A be an ideal of R. Prove that R/A is a field if A is maximal. 5

(d) Define factor ring. Let R be a ring and let A be a subring of R. Prove that the set of co-sets $\{r + A \mid r \in R\}$ is a ring under the operation

$(s + A) + (t + A) = (s + t) + A$ and

$(s + A)(t + A) = st + A$ if and only if A is an ideal of R. 1+5+4=10

(e) (i) Let ϕ be a ring homomorphism from R to S. Prove that the mapping from $R/ker\ \phi$ to $\phi(R)$, given by $r + ker\phi \to \phi(r)$ is an isomorphism. 5

(ii) Let R be a ring with unity and the characteristic of R is $n > 0$. Prove that R contains a sub-ring isomorphic to Z_n. If the characteristic of R is 0, then prove that R contains a sub-ring isomorphic to Z. 3+2=5

(f) Let F be a field and let $p(x) \in F[x]$. Prove that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

(g) Let F be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exists unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $deg\, r(x) \langle deg\, g(x)$. With the help of an example verify the division algorithm for $F[x]$. 7+3=10

(h) (i) If F is a field, then prove that $F[x]$ is a principal ideal domain. 5

(ii) Let F be a field and let $p(x)$, $a(x)$, $b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) | a(x)b(x)$, then prove that $p(x) | a(x)$ or $p(x) | b(x)$. 5

(i) Prove that every principal ideal domain is a unique factorization domain.

(j) (i) Prove that every Euclidean domain is a principal ideal domain. 5

(ii) Show that the ring

$$Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid ab \in Z\}$$

is an integral domain but not a unique factorization domain. 5
